

On the absolute Nörlund summability factors

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Abstract. *In this paper a theorem on the absolute Nörlund summability factors has been proved under more weaker conditions by using an almost increasing sequence.*

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1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the n -th Cesàro means of order α , with $\alpha > -1$, of the sequences (s_n) and (w_n) , respectively. The series $\sum a_n$ is said to be summable $|C, \alpha|$, if (see [4], [6])

$$\sum_{n=1}^{\infty} |u_n^\alpha - u_{n-1}^\alpha| = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha| < \infty. \quad (1)$$

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \dots + p_n \neq 0, \quad (n \geq 0) \quad (2)$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{\nu=0}^n p_{n-\nu} s_\nu \quad (3)$$

defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) . The series $\sum a_n$ is said to be summable $|N, p_n|$, if (see [7])

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty. \quad (4)$$

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In the special case where

$$p_n = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)\Gamma(n + 1)}, \quad \alpha \geq 0 \quad (5)$$

the Nörlund mean reduces to the (C, α) mean and $|N, p_n|$ summability becomes $|C, \alpha|$ summability. For $p_n = 1$ and $P_n = n$, we get the $(C, 1)$ mean and then $|N, p_n|$ summability becomes $|C, 1|$ summability. For any sequence (λ_n) we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \Delta\lambda_n - \Delta\lambda_{n+1}$. In [5] Kishore has proved the following theorem concerning $|C, 1|$ and $|N, p_n|$ summability methods.

Theorem 1. *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $|C, 1|$, then the series $\sum a_n P_n (n + 1)^{-1}$ is summable $|N, p_n|$.*

Ahmad [1] proved the following theorem for absolute Nörlund summability factors.

Theorem 2. *Let (p_n) be as in Theorem 1. If*

$$\sum_{\nu=1}^n \frac{1}{\nu} |t_\nu| = O(X_n) \quad \text{as } n \rightarrow \infty, \quad (6)$$

where (X_n) is a positive non-decreasing sequence and (λ_n) is a sequence such that

$$X_n \lambda_n = O(1), \quad (7)$$

$$n \Delta X_n = O(X_n), \quad (8)$$

$$\sum n X_n |\Delta^2 \lambda_n| < \infty, \quad (9)$$

then the series $\sum a_n P_n \lambda_n (n + 1)^{-1}$ is summable $|N, p_n|$.

Later on Bor [3] has proved Theorem 2. under weaker conditions in the following form.

Theorem 3. *Let (p_n) be as in Theorem 1. Let (X_n) be a positive non-decreasing sequence. If the conditions (6) and (7) of Theorem 2. are satisfied and the sequences (λ_n) and (β_n) are such that*

$$|\Delta \lambda_n| \leq \beta_n \quad (10)$$

$$\beta_n \rightarrow 0 \quad (11)$$

$$\sum n X_n |\Delta \beta_n| < \infty, \quad (12)$$

then the series $\sum a_n P_n \lambda_n (n + 1)^{-1}$ is summable $|N, p_n|$.

2. Main results

The aim of this paper is to prove *Theorem 3* under more weaker conditions. For this we need the concept of an almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [2]). Obviously, every increasing sequence is almost increasing but the converse need not be true, as can be seen from the example $b_n = ne^{(-1)^n}$. So we are weakening the hypotheses of the theorem replacing the increasing sequence by any almost increasing sequence. Now we shall prove the following theorem.

Theorem 4. *Let (p_n) be as in Theorem 1 and let (X_n) be an almost increasing sequence. If the conditions (6), (7), (10) and (12) of Theorem 2. and Theorem 3. are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.*

We need the following Lemma for the proof of our theorem.

Lemma 1. *Under the conditions on (X_n) , (λ_n) and (β_n) , as taken in the statement of the theorem, the following conditions hold, when (12) is satisfied:*

$$n\beta_n X_n = O(1) \text{ as } n \rightarrow \infty, \quad (13)$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty \quad (14)$$

Proof. Let $Ac_n \leq b_n \leq Bc_n$, where (c_n) is an increasing sequence. In this case

$$\begin{aligned} n\beta_n X_n &\leq nBc_n \left| \sum_{\nu=n}^{\infty} \Delta\beta_\nu \right| \leq nBc_n \sum_{\nu=n}^{\infty} |\Delta\beta_\nu| \leq B \sum_{\nu=n}^{\infty} \nu c_\nu |\Delta\beta_\nu| \\ &\leq (A/B) \sum_{\nu=n}^{\infty} \nu X_\nu |\Delta\beta_\nu| < \infty. \end{aligned}$$

Hence, $n\beta_n X_n = O(1)$ as $n \rightarrow \infty$.

Again

$$\begin{aligned} \sum_{n=1}^{\infty} \beta_n X_n &\leq B \sum_{n=1}^{\infty} c_n \beta_n = B \sum_{n=1}^{\infty} c_n \left| \sum_{\nu=n}^{\infty} \Delta\beta_\nu \right| \\ &\leq B \sum_{n=1}^{\infty} c_n \sum_{\nu=n}^{\infty} |\Delta\beta_\nu| = B \sum_{\nu=1}^{\infty} |\Delta\beta_\nu| \sum_{n=1}^{\nu} c_n \\ &\leq B \sum_{\nu=1}^{\infty} \nu c_\nu |\Delta\beta_\nu| \leq (B/A) \sum_{\nu=1}^{\infty} \nu X_\nu |\Delta\beta_\nu| < \infty. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \beta_n X_n < \infty$. □

Proof of Theorem 4. In order to prove the theorem, we need consider only the special case in which (N, p_n) is $(C, 1)$, that is, we shall prove that $\sum a_n \lambda_n$ is

summable $|C, 1|$. Our theorem will then follow by means of *Theorem 1*. Let T_n be the n -th $(C, 1)$ mean of the sequence $(na_n\lambda_n)$, that is,

$$T_n = \frac{1}{n+1} \sum_{\nu=1}^n \nu a_\nu \lambda_\nu \quad (15)$$

Using Abel's transformation, we have

$$\begin{aligned} T_n &= \frac{1}{n+1} \sum_{\nu=1}^n \nu a_\nu \lambda_\nu = \frac{1}{n+1} \sum_{\nu=1}^n \Delta \lambda_\nu (\nu+1) t_\nu + \lambda_\nu t_\nu \\ &= T_{n,1} + T_{n,2} \quad , \text{ say.} \end{aligned}$$

By (1), to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} (1/n) |T_{n,r}| < \infty \quad \text{for } r = 1, 2. \quad (16)$$

Now, we have

$$\begin{aligned} \sum_{n=2}^{m+1} (1/n) |T_{n,1}| &\leq \sum_{n=2}^{m+1} (1/n(n+1)) \left\{ \sum_{\nu=1}^{n-1} ((\nu+1)/\nu) |\Delta \lambda_\nu| |t_\nu| \right\} \\ &= O(1) \sum_{n=2}^{m+1} (1/n^2) \left\{ \sum_{\nu=1}^{n-1} \nu \beta_\nu |t_\nu| \right\} \\ &= O(1) \sum_{\nu=1}^m \nu \beta_\nu |t_\nu| \sum_{n=\nu+1}^{m+1} 1/n^2 = O(1) \sum_{\nu=1}^m \nu \beta_\nu |t_\nu| / \nu \\ &= O(1) \sum_{\nu=1}^{m-1} \Delta(\nu \beta_\nu) \sum_{r=1}^{\nu} |t_r| / r + O(1) m \beta_m \sum_{\nu=1}^m |t_\nu| / \nu \\ &= O(1) \sum_{\nu=1}^{m-1} |\Delta(\nu \beta_\nu)| X_\nu + O(1) m \beta_m X_m \\ &= O(1) \sum_{\nu=1}^{m-1} |\Delta \beta_\nu| \nu X_\nu + O(1) \sum_{\nu=1}^{m-1} |\beta_{\nu+1}| X_{\nu+1} + O(1) m \beta_m X_m \\ &= O(1) \quad \text{as } m \rightarrow \infty, \end{aligned}$$

by (6), (10), (12), (13) and (14).

Again

$$\begin{aligned} \sum_{n=1}^m (1/n) |T_{n,2}| &= \sum_{n=1}^m |\lambda_n| (|t_n| / n) \\ &= \sum_{n=1}^m |\Delta \lambda_n| \sum_{\nu=1}^n |t_\nu| / \nu + |\lambda_m| \sum_{n=1}^m |t_n| / n \\ &= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \\ &= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \rightarrow \infty, \end{aligned}$$

by (6), (7), (10) and (14). This completes the proof of the theorem. \square

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